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On best approximation of classes by radial functions

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Abstract

We investigate the radial manifolds $\mathcal{R}_n$ generated by a linear combination of n radial functions on $\mathbf{R}^d$. We consider the best approximation of function classes by the manifold $\mathcal{R}_n$. In particular, we prove that the deviation of the manifold $\mathcal{R}_n$ from the Sobolev class $W_2^{r,d}$ in the Hilbert space L_2 behaves asymptotically as $n^{-\frac{r}{d-1}}$. We show the connection between the manifold $\mathcal{R}_n$ and the space of algebraic polynomials $\mathcal{P}_{d,s}$ of degree s. Namely, we prove there exist constants c_1 and c_2 such that the space $\mathcal{P}_{d,s}$ is either contained or not in $\mathcal{R}_n$ as $n \geqslant c_1 s^{d-1}$ or $n < c_2 s^{d-1}$, respectively.

1. Introduction

In this work, we investigate properties of the manifold $\mathcal{R}_n$ of finite linear combinations of radial functions. The approximation of multivariable functions by the manifold $\mathcal{R}_n$ is also studied.

Let a be some point in the d-dimensional space $\mathbf{R}^d$. A radial function with the fixed center a is defined as a function on $\mathbf{R}^d$ of the form $g_a(x) = g(|x-a|^2)$, where $x \in \mathbf{R}^d$, $g: \mathbf{R} \to \mathbf{R}$, and the quantity $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ is the Euclidean norm of the point x.

Let $\mathcal{A}$ be some subset in $\mathbf{R}^d$. Consider the linear space of radial functions

$$\mathcal{R}(\mathcal{A}) = span\{g(|x-a|^2):\ a \in \mathcal{A},\ g \in C(\mathbf{R})\}, \tag{1}$$

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where a runs over the set $\mathscr{A}$ and g is any continuous functions on $\mathbf{R}$. For a fixed natural number n consider the subset in $\mathscr{R}(\mathscr{A})$:

$$\mathscr{R}_n = \bigcup \{\mathscr{R}(\mathscr{A}) \colon \; card\, \mathscr{A} \leqslant n\},$$

which is the union of all sets $\mathscr{R}(\mathscr{A})$, where $\mathscr{A}$ runs over all possible subsets in $\mathbf{R}^d$ of cardinality at most n. Note that unlike the space $\mathscr{R}(\mathscr{A})$, the set $\mathscr{R}_n$ is not a linear space. We will therefore call $\mathscr{R}_n$ a radial manifold.

Let D be a compact set in the space $\mathbf{R}^d$. Consider the Hilbert space $L_2(D)$ of square-integrable functions defined on D and norm

$$||f||_{L_2(D)} = \left(\int_D |f|^2 \, dx \right)^{1/2}.$$

We denote the ball of radius a in $\mathbf{R}^d$ by $B^d(a) = \{x = (x_1, \dots, x_d) \colon \; \sum_{i=1}^d x_i^2 \leqslant a^2\}$. In the sequel we will mainly consider the unit ball $B^d(1)$. We simplify the notation somewhat by setting $B^d = B^d(1)$ and $L_2 = L_2(B^d)$.

For any two sets $W, H \subset L_2$ we define the distance of H to W by

$$dist(W, H, L_2) = \sup_{f \in W} dist(f, H, L_2),$$

where $dist\ (f, H, L_2) = \inf_{h \in H} ||f - h||_{L_2}$.

Let $\rho = (\rho_1, \dots, \rho_d)$ be a multi-index vector, that is, ρ is the vector with non-negative integer coordinates, $|\rho| = \rho_1 + \cdots + \rho_d$. Introduce the differential operator $\mathscr{D}^\rho = \partial^{|\rho|}/\partial^{\rho_1} x_1 \dots \partial^{\rho_d} x_d$. Let r be any natural number. In the space L_2 we consider the Sobolev class of functions

$$W_2^{r,d} \coloneqq \{f \colon \; ||f||_{W_2^{r,d}} \leqslant 1\},$$

where the norm is defined as

$$||f||_{W_2^{r,d}} = ||f||_{L_2} + \max_{|\rho|=r} ||D^\rho f||_{L_2}.$$

Let c, and $c_1, c_2, \dots$ be positive constants depending solely on the parameters r and d. For two positive sequences a_n and b_n, $n = 0, 1, \dots$ we write $a_n \asymp b_n$ if there exist positive constants c_1 and c_2 such that $c_1 \leqslant a_n / b_n \leqslant c_2$ for all $n = 0, 1, \dots$.

In [20] a similar result was proven about best approximation by ridge functions. If $\mathscr{H}_n$ is the manifold consisting of arbitrary linear combinations of n ridge functions, that is

$$\mathscr{H}_n \coloneqq \left\{ \sum_{k=1}^n g_k(x \cdot a_k) \colon \; a_k \in \mathbf{R}^d, \;\; g_k \in C(\mathbf{R}) \right\},$$

then the deviation of the Sobolev class $W_2^{r,d}$ from $\mathscr{H}_n$ satisfies the asymptotics

$$dist(W_2^{r,d}, \mathscr{H}_n, L_2) \asymp n^{-\frac{r}{d-1}}. \tag{2}$$

A series of works on the density of ridge and radial manifolds in functional spaces is considered by Agranovsky and Quinto [3], Agranovsky et al. [2], Lin and Pinkus [16,17], Pinkus [36]. The approximation properties of ridge manifolds were studied in

Barron [4], DeVore et al. [13], Maiorov [20], Maiorov and Meir [22]. Makovoz [27], Mhaskar and Michelli [29], Mhaskar [28], Oskolkov [32], Petrushev [33], Pinkus [37], Temlyakov [46]. In Gordon et al. [14] the results about best approximation by ridge functions in the Banach space L_p are considered. See also Pinkus [38] for a review of this theme.

The approximation of functions by radial waves was considered by Buhmann [7,8], Light and Wayne [15], Mhaskar et al. [30], Pinkus [35], Bumann, Dyn and Levin [11], Schaback [41,42], Bejancu [6], Maiorov [21].

2. Main results

The main part of this paper investigates approximations of functions in the Sobolev class using a manifold of radial functions $\mathscr{R}_n$. We obtain upper and lower bounds, which are asymptotically equal, on the deviation $dist(W_2^{r,d}, \mathscr{R}_n, L_2)$ of the Sobolev class from the manifold $\mathscr{R}_n$ in the space L_2.

We will prove that an upper bound for this deviation is attained by a manifold of radial functions $\mathscr{R}(\mathscr{A}_n)$ with some fixed collection of n center points independent of the approximating function. That is, for every n there exists a collection of points $\mathscr{A}_n = \{a_1, \dots, a_n\}$ in $\mathbf{R}^d$ such that the manifold

$$\mathscr{R}(\mathscr{A}_n) = \mathscr{R}(a_1, \dots, a_n) = span\{g(|x - a_i|^2)\colon\ i = 1, \dots, n,\ \ g \in C(\mathbf{R})\}$$

realizes the optimal estimate

$$dist(W_2^{r,d}, \mathscr{R}_n, L_2) \asymp dist(W_2^{r,d}, \mathscr{R}(\mathscr{A}_n), L_2).$$

Note that if we take the points $a_1, \dots, a_n$ sufficiently large in modulus then the upper bound in the approximation problem is easily reduced to the upper bound in approximation by ridge functions. That is, let g be any continuous function on $\mathbf{R}$ and a be a fixed non-zero vector in $\mathbf{R}^d$. We set $\bar{a} = -\frac{2a}{|a|}$ and introduce the function $\varepsilon_a(x) = \frac{|x|^2}{|a|}$. Then the radial function may be rewritten as

$$\begin{aligned} g(|x - a|^2) &= g(|x|^2 - 2x \cdot a + |a|^2) = g(|a|(\varepsilon_a(x) + x \cdot \bar{a} + |a|)) \\ &= h(x \cdot \bar{a} + \varepsilon_a(x)), \end{aligned}$$

where h is some continuous function on $\mathbf{R}$. Since for sufficiently large $|a|$ the function $\varepsilon_a(x)$ is sufficiently small on the unit ball B^d then the function $h(x \cdot \bar{a} + \varepsilon_a(x))$ and hence the function $g(|x - a|^2)$ is almost a ridge function. Thus by using approximation by ridge functions (see (2)) we obtain this next claim: for any natural n there exist vectors $a_1, \dots, a_n$ with a sufficiently large modulo in $\mathbf{R}^d$ such that the following estimate is true:

$$dist(W_2^{r,d}, \mathscr{R}(a_1, \dots, a_n), L_2) \leqslant cn^{-\frac{r}{d-1}}, \tag{3}$$

where c is dependent only on r and d. We will also show that for the optimal approximation of the class $W_2^{r,d}$ we can also take center points $a_1, \dots, a_n$ not of a large modulus. In the next theorem we embark upon two problems. We show that for obtaining estimate (3) it suffices to take the centers $a_1, \ldots, a_n$ on the unit sphere. We also prove that these points are asymptotically optimal points for the approximation of the Sobolev class by the manifold $\mathcal{R}_n$.

Theorem 2.1. *Let $d \geqslant 2$, $r > 0$ and n be any natural numbers. Then*

1. *there exist points $a_1, \ldots, a_n$ on the unit sphere S^{d-1} such that*

$$dist(W_2^{r,d}, \mathcal{R}(a_1, \ldots, a_n), L_2) \leqslant c_1 n^{-\frac{r}{d-1}},$$

2. *for any points $a_1, \ldots, a_n$ in the space $\mathbf{R}^d$ the following lower estimate is true*

$$dist(W_2^{r,d}, \mathcal{R}(a_1, \ldots, a_n), L_2) \geqslant c_2 n^{-\frac{r}{d-1}}.$$

Here c_1 and c_2 are dependent only on r and d.

Note that Theorem 2.1 can also be extended to general compact domains D by use of standard extension theorems, as in [1].

We add two consequences resulting from Theorem 2.1.

Let s be a natural number. Consider the space

$$\mathcal{P}_{d,s} = span\{x_1^{k_1} \cdots x_d^{k_d} \colon |k| = k_1 + \cdots + k_d \leqslant s\}$$

consisting of all algebraic polynomials on $\mathbf{R}^d$ with real coefficients of degree at most s. Denote by $\mathcal{P}_{d,s}^{\mathrm{hom}}$ the subspace of $\mathcal{P}_{d,s}$ consisting of all homogeneous polynomials of degree s, i.e., $\mathcal{P}_{d,s}^{\mathrm{hom}} = span\{x_1^{k_1} \cdots x_d^{k_d} : |k| = s\}$. Let $B\mathcal{P}_{d,s} = \{p \in \mathcal{P}_{d,s} : ||p||_{L_2} \leqslant 1\}$ be the unit ball in the space $\mathcal{P}_{d,s}$.

Construct the class $\mathcal{B}_2^{d,r}(b)$ consisting of all functions $f \in L_2(B^d)$ representable in the form

$$f(x) = \sum_{s=0}^{\infty} p_s(x), \quad p_s \in b2^{-rs}\, B\mathcal{P}_{d,s}, \quad s = 0, 1, ..,$$

where the series converges in the L_2 sense. We set $\mathcal{B}_2^{d,r} = \mathcal{B}_2^{d,r}(1)$. The class $\mathcal{B}_2^{d,r}$ consists of all functions which may be approximated by the polynomial space $\mathcal{P}_{d,s}$ at a rate of order s^{-r}.

It follows from Jackson's Theorem that the Sobolev class $W_2^{r,d}$ belongs to the class $\mathcal{B}_2^{d,r}(b)$ with some constant b. However the inverse embedding is not true, i.e., the class $\mathcal{B}_2^{d,r}$ does not belong to $W_2^{r,d}(b_1)$ for any positive constant b_1. Nevertheless for the class $\mathcal{B}_2^{d,r}(b)$ the same result as for the class $W_2^{r,d}$ is true, that is, the deviation of $\mathcal{B}_2^{d,r}(b)$ from the manifold $\mathcal{R}_n$ satisfies the same asymptotic.

Consequence 2.2. *Let $d \geqslant 2$, $r > 0$, and n be any natural number. Then*

1. *there exist points $a_1, \ldots, a_n$ on the unit sphere S^{d-1} such that*

$$dist(\mathscr{B}_2^{d,r}, \mathscr{R}(a_1, \ldots, a_n), L_2) \leqslant c_1 n^{-\frac{r}{d-1}},$$

2. *for any points $a_1, \ldots, a_n$ in the space $\mathbf{R}^d$ the following lower estimate is true*

$$dist(\mathscr{B}_2^{d,r}, \mathscr{R}(a_1, \ldots, a_n), L_2) \geqslant c_2 n^{-\frac{r}{d-1}}.$$

The next consequence contains information about the structure of the radial manifold $\mathscr{R}_n$.

Consequence 2.3. *Let s be any natural number. Consider the spaces $\mathscr{P}_{d,s}$ of polynomials of degree $\leqslant s$. Then there exist positive constants $0 < c < 1$, $c' > 1$ and c_3 dependent only on d, such that*

1. *if $n \geqslant cs^{d-1}$ then $\mathscr{P}_{d,s}$ belongs to the manifold $\mathscr{R}(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are some points on the sphere S^{d-1},*
2. *if $n \leqslant c' s^{d-1}$, then $\mathscr{P}_{d,s}$ does not belong to $\mathscr{R}_n$, and the following inequality is true*

$$dist(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2) \geqslant c_3 > 0.$$

We describe briefly the proof of Theorem 2.1. This theorem consists of two parts: the upper and lower bounds. In Section 3, we construct an orthogonal basis for the algebraic polynomials $\Pi = \{P_i\}_{i=0}^{\infty}$ in the space $L_2(B^d)$. In Sections 4–6, we investigate the moments of radial functions with centers at a point a relative to the basis Π, that is

$$b_i(g_a) = \int_{B^d} g(|x - a|) P_i(x)\, dx, \quad a \in \mathbf{R}^d.$$

We prove that for the moments $b_i(g_a)$ for any i the formula of partition of variables g and a holds, that is, any moment $b_i(g_a)$ may be given as a finite linear combination of functions of the form $\gamma(g)\pi(a)$, where γ and π are functions depending only on g and the vector a, respectively. By the help of the formulas of partition of variables we construct some finite dimensional linear system of equations relative to the unknown moments of the initial approximated function from the manifold $\mathscr{R}_n$. In Section 7 we prove the upper bound in Theorem 2.1. The lower bound in Theorem 2.1 is proved in Sections 8 and 9. The proof of the lower bound is based on comparing the entropy numbers of the polynomial ball $B\mathscr{P}_{d,s}$ and the radial manifold ball $B\mathscr{R}_n = \{r \in \mathscr{R}_n : ||r||_{L_2} \leqslant 1\}$.

3. Orthogonal system of algebraic polynomials on the ball

In this section, we construct special orthonormal systems of polynomials on the unit ball B^d. Orthogonal systems of polynomials on the ball play the important part in problems of approximations of multivariable functions by manifolds of linear combinations of ridge functions (the plane waves) (see [13,18,33]). In the works of [13,33] methods were developed for the construction of orthogonal projections on polynomial subspaces and approximation by ridge functions. The system of Gegenbauer orthogonal polynomials is the main tool used in the construction of orthogonal systems of polynomials on the ball [18]. Note, in particular, that in two dimensions, $d = 2$, these Gegenbauer polynomials coincide with Chebyshev polynomials.

In our work the system of orthogonal polynomials on the unit ball is obtained, in a sense, by the convolution of two orthogonal systems. These are the system of Gegenbauer polynomials on the segment $[-1, 1]$, and the system of spherical harmonics on the unit sphere S^{d-1}. We describe this construction next.

Let $L_2(S^{d-1})$ be the Hilbert space consisting of all the complex-valued square-integrable functions $h(\xi)$ on the sphere S^{d-1} with the inner product

$$(h_1, h_2) = \int_{S^{d-1}} h_1(\xi)\bar{h}_2(\xi)\, d\xi\, d\xi, \quad h_1, h_2 \in L_2(S^{d-1}),$$

where by $d\xi$ we denote the normalized Lebesgue measure on the sphere S^{d-1}.

In the space $L_2(S^{d-1})$ consider (see the appendix) the subspace H consisting of the restrictions on S^{d-1} of the harmonic functions on $\mathbf{R}^d$. Let H_s be the subspace in H generated by all spherical harmonics of degree at most s, i.e. all harmonic polynomials of degree at most s. Let H_s^{hom} be the subspace of H_s formed by all homogeneous spherical harmonics of degree s. The functions $\{h_{sk}\}_{k \in K^s}$ generate a basis in the space H_s^{hom} (see the appendix).

The space $H_s = H_0^{\text{hom}} \oplus H_1^{\text{hom}} \oplus \cdots \oplus H_s^{\text{hom}}$ is the direct sum of the orthogonal subspaces of the spherical harmonics of degrees $0, 1, \ldots, s$. Denote by N_s the dimension of the space H_s. We have $N_s \asymp s^{d-1}$. Indeed, as shown in (A.4), using the relation $\dim H_s^{\text{hom}} \asymp s^{d-2}$ we obtain

$$N_s = \dim H_s = \dim H_0^{\text{hom}} + \dim H_1^{\text{hom}} + \cdots + \dim H_s^{\text{hom}} \asymp s^{d-1}.$$

In the space H we introduce the family of functions $\mathscr{B} \equiv \mathscr{B}(S^{d-1}) \equiv \{h_i\}_{i=0}^{\infty}$ consisting (see (A.2)) of all ordered spherical harmonics, that is, the functions

$$\bigcup_{s=0}^{\infty} \{h_{s,k}\}_{k \in K^s}.$$

The set $\mathscr{B}(S^{d-1})$ is an orthonormal basis in the space H, i.e., for indices $i \neq i'$ we have $(h_i, h_{i'}) = \delta_{ii'}$, where $\delta_{ii'} = 0$ for $i \neq i'$, and $\delta_{ii} = 1$.

As shown in the appendix, consider next the Gegenbauer polynomials $C_n^{d/2}(t)$, $t \in \mathbf{R}$, of degree n associated with $d/2$. We normalize the polynomial $C_n^{d/2}$ by a factor, i.e., we set

$$u_n(t) = v_n^{-1/2} C_n^{d/2}(t), \quad \text{where } v_n = \frac{\pi^{1/2}(d)_n \Gamma(\frac{d+1}{2})}{(n+d/2)n!\Gamma(d/2)},$$

where $(a)_0 = 1$, and $(a)_n = a(a+1)\cdots(a+n-1)$.

Let i and j be two arbitrary indices from $\mathbf{Z}_+$. Construct on $\mathbf{R}^d$ the function

$$P_{ij}(x) = v_j \int_{S^{d-1}} h_i(\xi) u_j(x \cdot \xi)\, d\xi, \quad \text{where } v_j = \left(\frac{(j+1)_{d-1}}{2(2\pi)^{d-1}} \right)^{1/2}, \tag{4}$$

and $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$ is the inner product of the vectors x and ξ.

From (4) we see that for any $i,j \in \mathbf{Z}_+$ the function P_{ij} is a polynomial on $\mathbf{R}^d$ of degree j. Note that if the indices i and j are such that the degrees of the polynomials h_i and u_j satisfy the inequality $deg\, h_i > deg\, u_j = j$, then $P_{ij}(x) \equiv 0$ (see [43, (A.10)]).

For a given integer j we let $\varepsilon_j = 0$ if j is even integer, and $\varepsilon_j = 1$ if j is an odd integer. Consider the set of matching indices

$$I = \{(i,j) : j \in \mathbf{Z}_+,\ deg\, h_i \in \{j, j-2, \ldots, \varepsilon_j\}\}. \tag{5}$$

Note that each matching index $(i,j) \in I$ satisfies the condition: the parity of the degree of h_i coincides with the parity of the degree of the polynomial u_j. Construct the system of polynomials

$$\Pi := \Pi(B^d) := \{P_{ij}\}_{(i,j)\in I}. \tag{6}$$

Lemma 3.1. *The set $\Pi(B^d)$ of polynomials is a complete orthonormal system of functions in the space $L_2(B^d)$.*

Proof. Orthonormality of the polynomial system $\Pi(B^d)$ was proved by Maiorov [20]. Therefore for any of the matching indices (i,j), $(i',j') \in I$ the following relation holds:

$$\langle P_{ij}, P_{i',j'} \rangle := \int_{B^d} P_{ij}(x) \bar{P}_{i'j'}(x)\, dx = \delta_{ii'}\delta_{j'j'}. \tag{7}$$

We show that the set $\Pi(B^d)$ is a complete system in the space $L_2(B^d)$. By Weierstrass' Theorem it is sufficient to prove that for any natural s the subspace of polynomials $\mathscr{P}_{d,s}$ of degree s coincides with the subspace

$$span\{P_{ij} \in \Pi(B^d):\ deg P_{ij} \leqslant s\}$$

or the space of homogeneous polynomials $\mathscr{P}_{d,s}^{\mathrm{hom}}$ of degree s coincides with the subspace $\mathrm{span}\{P_{is} \in \Pi(B^d) : deg\, h_i \in \{s, s-2, \ldots, \varepsilon_s\}\}$.

Let H_l^{hom} be the space of homogeneous harmonic polynomials of degree l. It is known [43, Chapter 4, Section 2] that any polynomial $p \in \mathscr{P}_{d,s}^{\mathrm{hom}}$ may be represented by

$$p(x) = p_0(x) + |x|^2 p_1(x) + \cdots + |x|^{2l} p_l(x),$$

where $p_k \in H^{\text{hom}}_{s-2k}$, $k = 0, 1, \dots, l$; $l = (s - \varepsilon_s)/2$. From here it follows that the dimension of the space $\mathscr{P}^{\text{hom}}_{d,s}$ is equal to

$$dim\, \mathscr{P}^{\text{hom}}_{d,s} = dim\, H^{\text{hom}}_s + dim\, H^{\text{hom}}_{s-2} + \cdots + dim\, H^{\text{hom}}_{\varepsilon_s}.$$

We have $dim\, H^{\text{hom}}_j = card\{i \in \mathbf{Z}_+ : deg\, h_i = j\}$, and $deg P_{ij} = j$. Therefore

$$\begin{aligned} dim\, \mathscr{P}^{\text{hom}}_{d,s} &= \sum_{j \in \{s, s-2, \dots, \varepsilon_s\}} card\{i : deg\, h_i = j\} \\ &= card\{i : deg\, h_i \in \{s, s-2, \dots, \varepsilon_s\}\} \\ &= card\{P_{is} \in \Pi : deg\, h_i \in \{s, s-2, \dots, \varepsilon_s\}\}. \end{aligned}$$

Hence by the orthogonality property (7) of the polynomials P_{is} we obtain

$$\mathscr{P}^{\text{hom}}_{d,s} = span\{P_{is} \in \Pi : deg\, h_i \in \{s, s-2, \dots, \varepsilon_s\}\}.$$

Hence the lemma is proved. □

We insert in the set I the subset of matching indices $I_s = \{(i,j) \in I : j \leqslant s\}$, and we consider in the function system Π the finite subsystem $\Pi_s = \{P_{ij}\}_{(i,j) \in I_s}$. From Lemma 3.1 we directly obtain the next statement.

Consequence 3.2. *The polynomial set Π_s is an orthonormal basis in the space of polynomials $\mathscr{P}_{d,s}$.*

4. The moments of radial functions relative to the basis $\Pi(B^d)$ and the orthogonal groups of rotations

Let f be any function from the space $L_2(B^d)$. From Lemma 3.1 we can decompose the function in its orthogonal series by the system $\Pi(B^d) = \{P_{ij}\}_{(i,j) \in I}$:

$$f(x) = \sum_{(i,j) \in I} b_{ij} P_{ij}(x), \quad b_{ij} = \langle f, P_{ij} \rangle.$$

We denote the coefficients b_{ij} of this decomposition as the moments of the function f relative to the basis $\Pi(B^d)$.

Let a be a fixed point on the unit sphere S^{d-1}. Consider a radial function $g_a(x) = g(|x-a|^2)$, $x \in B^d$, with center a and $g \in C(\Delta)$ being on the segment $\Delta = [0, 4]$.

In this section, we show that for any matching indices $(i,j) \in I$ the moments $b_{ij}(g_a)$ of the radial function g_a may be represented as a linear combination of the moments $\{b_{i'j}(g_e)\}$, $i' = 0, 1, \dots$, with a unique center at the point $e = (0, \dots, 0, 1) \in S^{d-1}$.

The case $d = 2$: We first consider the two-dimensional case. Let ξ be any point on S^1 with the coordinates $(\cos \tau_\xi, \sin \tau_\xi)$. Set $t_i(\tau) = \exp(-\sqrt{-1} i \tau)$. Then the spherical harmonics equal $h_i(\xi) = t_i(\tau_\xi)$, $i = 0, \pm 1, \dots$. Therefore the system $\Pi(B^2)$ of functions consists of polynomials of the form

$$P_{ij}(x) = v_j \int_{S^1} h_i(\xi) u_j(x \cdot \xi)\, d\xi, \quad j \in \mathbf{Z}_+, \quad i \in \{\pm j, \pm (j-2), \ldots, \pm \varepsilon_j\}.$$

Introduce the orthogonal matrix

$$A = \begin{pmatrix} \cos \tau_a & -\sin \tau_a \\ \sin \tau_a & \cos \tau_a \end{pmatrix}, \quad a \in S^1$$

translating the point $e = (0, 1)$ to the point a.

For the moments b_{ij} of the radial function $g_a(x)$ we have

$$\begin{aligned} b_{ij}(g_a) &= v_j \int g(|x-a|^2) \bar{P}_{ij}(x)\, dx \\ &= v_j \int_{S^1} \bar{h}_i(\xi)\, d\xi \int_{B^2} g(|x-a|^2) u_j(x \cdot \xi)\, dx \\ &= v_j \int_{S^1} \bar{h}_i(A\xi)\, d\xi \int_{B^2} g(|x-e|^2) u_j(x \cdot \xi)\, dx. \end{aligned}$$

Since $\bar{h}_i(A\xi) = \bar{t}_i(\tau_\xi - \tau_a) = \bar{t}_i(\tau_\xi) t_i(\tau_a) = \bar{h}_i(\xi) t_i(\tau_a)$, then we obtain

$$\begin{aligned} b_{ij}(g_a) &= v_j t_i(\tau_a) \int_{S^1} \bar{h}_i(\xi)\, d\xi \int_{B^2} g(|x-e|^2) u_j(x \cdot \xi)\, dx \\ &= v_j t_i(\tau_a) \int_{B^d} g(|x-e|^2) \bar{P}_{ij}(x)\, dx = t_i(\tau_a)\, b_{ij}(g_e). \end{aligned}$$

Thus the following lemma is valid.

Lemma 4.1. *Let $a = (\cos \tau_a, \sin \tau_a)$ be any point on the circumference S^1. Then the moments of the radial function g_a are equal to*

$$b_{ij}(g_a) = t_i(\tau_a) b_{ij}(g_e) \quad (i,j) \in I.$$

The case $d > 2$: For the proof of the result in the general case, i.e. $d \geqslant 2$, we will need some facts from the theory of representation of orthogonal groups. Consider the group of orthogonal rotations $SO(d)$ in the space $\mathbf{R}^d$, that is, the set of all square real matrices A of order d with determinant $det\, A = 1$.

Let $T = \{T(A)\}$ be the infinite-dimensional representation of the group $SO(d)$ in the Hilbert space $L_2(S^{d-1})$ of functions on the unit sphere S^{d-1}. The representation T is the mapping of the group $SO(d)$ to the set of linear operators on the space $L_2(S^{d-1})$ of the form

$$(T(A)h)(x) = h(Ax), \quad h \in L_2(S^{d-1}).$$

The mapping T is a group homomorphism, that is, satisfies the group relations: $T(AB) = T(A)T(B)$ and $T(A^{-1}) = T^{-1}(A)$, for any $A, B \in SO(d)$.

Consider in the space $L_2(S^{d-1})$ a basis $\mathcal{B} = \{h_i\}_{i=0}^{\infty}$ consisting of all the spherical harmonics introduced in Section 3. It is known [48] that the representation T in the basis $\mathcal{B}$ may be described as a collection of infinite dimensional matrices, that is, to any matrix $A \in SO(d)$ there corresponds the infinite dimensional matrix

$$T(A) = (t_{ii'}(A))_{i,i'=0}^{\infty},$$

and $T(A)$ satisfies the conditions:

1. If $h(\xi) = \sum_{i=0}^{\infty} c_i h_i(\xi)$ is the arbitrary function in the space $L_2(S^{d-1})$ then

$$(T(A)h)(\xi) = h(A\xi) = \sum_{i=0}^{\infty} \left(\sum_{i'=0}^{\infty} t_{ii'}(A) c_{i'} \right) h_i(\xi). \tag{8}$$

2. The set $\{t_{ii'}(A)\}_{i,i'=0}^{\infty}$ is a linearly independent system of functions on $SO(d)$.

In particular from (8) for any the spherical harmonic h_i we have

$$h_i(A\xi) = \sum_{i'=0}^{\infty} t_{ii'}(A) h_{i'}(\xi). \tag{9}$$

It is known (cf. [48]) that the representation T is invariant in every subspace H_s^{hom} of the homogeneous harmonic polynomials of degree s. That is, for any matrix $A \in SO(d)$ and any polynomial $h \in H_s^{\mathrm{hom}}$ the polynomial $h(Ax)$ also belongs to the subspace H_s^{hom}. Therefore any infinite-dimensional matrix $T(A)$ is the block matrix in which the square matrices $T_s(A)$ of order $dim\, H_s^{\mathrm{hom}}$ stand on the main diagonal, i.e.,

$$T(A) = T_0(A) \oplus \cdots \oplus T_s(A) \oplus \cdots.$$

Note that for every $s = 0, 1, \ldots$ the matrix set $\{T_s(A) : A \in SO(d)\}$ is the finite-dimensional representation of group $SO(d)$ in the space H_s^{hom}, that is, it satisfies conditions 1 and 2.

Lemma 4.2. *Let a be any point on the unit sphere S^{d-1}. Define the matrix $A \in SO(d)$ such that $Ae = a$, $e = (0, \ldots, 0, 1)$. Let $g(t)$ be any continuous function defined on $[0, 4]$, and $g_a(x) = g(|x-a|^2)$ be the radial function with center a. Then for any matching index (i, j) from the set I*

$$b_{ij}(g_a) = \sum_{i'=0}^{\infty} t_{ii'}(A) b_{i'j}(g_e).$$

Proof. By Definition (4), using twice the invariant property of the Lebesgue measure relative to rotation in the space $\mathbf{R}^d$ we have

$$\begin{aligned}\langle g_a, P_{ij}\rangle &= v_j \int_{B^d} g(|x-a|^2)\,dx \int_{S^{d-1}} \bar{h}_i(\xi)u_j(x\cdot\xi)\,d\xi\\ &= v_j \int_{B^d} g(|x-e|^2)\,dx \int_{S^{d-1}} \bar{h}_i(\xi)u_j(Ax\cdot\xi)\,d\xi\\ &= v_j \int_{B^d} g(|x-e|^2)\,dx \int_{S^{d-1}} \bar{h}_i(A\xi)u_j(x\cdot\xi)\,d\xi.\end{aligned}$$

Applying (9) we obtain

$$\begin{aligned}&\int_{B^d} g(|x-e|^2)\,dx \int_{S^{d-1}} \bar{h}_i(A\xi)u_j(x\cdot\xi)\,d\xi\\ &\quad= \sum_{i'=0}^{\infty} t_{ii'}(A) \int_{B^d} g(|x-e|^2)\,dx \int_{S^{d-1}} \bar{h}_{i'}(\xi)u_j(x\cdot\xi)\,d\xi\\ &\quad= \sum_{i'=0}^{\infty} t_{ii'}(A) \int_{B^d} g(|x-e|^2)P_{i'j}(x)\,dx = \sum_{i'=0}^{\infty} t_{ii'}(A)b_{i'j}(g_e).\end{aligned}$$

The lemma is proved. □

5. A partition of variables for the moments of radial functions with centers on S^{d-1}

Let $g_a(x) = g(|x-a|^2)$ be any radial function with center $a\in S^{d-1}$, and let P be any polynomial on $\mathbf{R}^d$. In this section, we show that every moment of the function g_a relative to the polynomial P may be represented by a finite linear combination of functions taking the form of a product of a function of g and a function of a. That is, every moment of the function g_a is, in some sense, given by a partition of variables g and a.

Theorem 5.1. *Let $g\in C(\Delta)$, $\Delta = [0,4]$, and $a\in S^{d-1}$. Let P be any polynomial on $\mathbf{R}^d$ of degree s, $s\geqslant d^2/2$. Set $v = 2d+5$. Then*

$$\langle g_a, P\rangle = \sum_{m=0}^{vs} \pi_m(a;P)\,\gamma_m(g),$$

where the functions $\pi_m(a;P)$ are some polynomials on the vector a of degree s, and the $\gamma_m(g)$ are some linear functionals on $C(\Delta)$.

Proof. Let $A\in SO(d)$ be the orthogonal matrix such that $Ae = a$, $e = (0,\dots,0,1)$. Then by the invariance of measure dx relative to rotation in $\mathbf{R}^d$

we have

$$\begin{aligned}
\langle g_a, P \rangle &= \int_{B^d} g(|x-a|^2)P(x)\,dx \\
&= \int_{B^d} g(|x-e|^2)P(Ax)\,dx \\
&= \int_{B^d-e} g(|x|^2)P(A(x+e))\,dx \\
&= \sum_{|l|\leqslant s} \pi_l(A;P) \int_{B^d-e} g(|x|^2)x^l\,dx, \qquad (10)
\end{aligned}$$

where $l=(l_1,\dots,l_d)$ is the multi-index, $|l|=l_1+\cdots+l_d$, $x^l=x_1^{l_1}\cdots x_d^{l_d}$, and $\pi_l(A;P)$ are some polynomials of degree $|l|$ on the d^2 elements of the matrix A.

Introduce in the space $\mathbf{R}^d$ the spherical system of coordinates:

$$\begin{aligned}
x_1 &= r\sin\omega_{d-1}\dots\sin\omega_2\sin\omega_1 := ru_1(\omega), \\
x_2 &= r\sin\omega_{d-1}\dots\sin\omega_2\cos\omega_1 := r\,u_2(\omega), \\
&\vdots \\
x_{d-1} &= r\sin\omega_{d-1}\cos\omega_{d-2} := r\,u_{d-1}(\omega), \\
x_d &= r\cos\omega_{d-1} := r\,u_d(\omega),
\end{aligned}$$

where the variables vary as follows: $r\geqslant 0$, $0\leqslant\omega_1<2\pi$, $0\leqslant\omega_k<\pi$, $k=2,\dots,d-1$. We denote this formulas concisely as

$$x = ru(\omega), \quad u(\omega)=(u_1(\omega),\dots,u_d(\omega)). \qquad (11)$$

The Lebesgue measure in the spherical coordinates is

$$\begin{aligned}
dx = dx_1\dots dx_d &= c_d r^{d-1}\sin^{d-2}\omega_{d-1}\cdot\ldots\cdot\sin\omega_2\,d\omega_1\dots d\omega_{d-1}\,dr \\
&:= r^{d-1}v(\omega)\,d\omega\,dr,
\end{aligned}$$

where $c_d=\Gamma(d/2)/(2\pi)^{d/2}$.

Denote by $S^{d-1}(r)=\{x\in\mathbf{R}^d : |x|=r\}$ the sphere in $\mathbf{R}^d$ of radius r. We consider the surface

$$\Omega(r) = S^{d-1}(r)\cap(B^d-e)$$

in the ball $B^d-e=\{x-e : x\in B^d\}$. Clearly, $B^d-e=\cup_{0\leqslant r\leqslant 2}\Omega(r)$ and $\Omega(r)\cap\Omega(r')=\emptyset$ for any $r\neq r'$.

Now fix r. We represent the surface $\Omega(r)$ by the spherical system of coordinates. In the rectangular system of coordinates the surface $\Omega(r)$ is described by the system consisting of the equation and the inequality

$$x_1^2+\cdots+x_d^2=r^2, \quad x_1^2+\cdots+x_{d-1}^2+(x_d-1)^2\leqslant 1.$$

It follows from the definition of the spherical system of coordinates that the points on the surface $\Omega(r)$ satisfy the inequality $r^2\leqslant 2x_d=2r\cos\omega_{d-1}$, that is

$$r\leqslant 2\cos\omega_{d-1}, \quad \textit{or} \quad 0\leqslant\omega_{d-1}\leqslant\arccos(r/2).$$

Set $\alpha_r = r/2$. Then the surface $\Omega(r)$ in the spherical system of coordinates is

$$\Omega(r) = \{(r, \omega) : (\omega_1, \dots, \omega_{d-1}) \in [0, 2\pi] \times [0, \pi]^{d-3} \times [0, \arccos(r/2)]\}. \tag{12}$$

We may represent the last integral in (10) for fixed l as

$$\begin{aligned} I_l &:= \int_{B^d - e} g(|x|^2) x^l \, dx = \int_{B^d - e} g(|x|^2) x^{l_1} \cdots x^{l_d} \, dx \\ &= \int_0^2 g(r^2) r^{|l|+d-1} \, dr \int_{\Omega(r)} u_1^{l_1}(\omega) \cdots u_d^{l_d}(\omega) v(\omega) \, d\omega. \end{aligned} \tag{13}$$

The computation of the last integral is done next.

Lemma 5.2. *Let $s \geqslant d^2/2$. Then for any multi-index l with $|l| \leqslant s$ the following equality holds*:

$$\begin{aligned} &\int_{\Omega(r)} u_1^{l_1}(\omega) \cdots u_d^{l_d}(\omega) v(\omega) \, d\omega \\ &\quad = \delta_l \arccos \alpha_r + \gamma_l + P_{l,s}(\alpha_r) + \sqrt{1 - \alpha_r^2}\, Q_{l,s}(\alpha_r), \end{aligned}$$

where $\alpha_r = r/2$, and δ_l, γ_l are some constants, and $P_{l,s}$, $Q_{l,s}$ are some polynomials of degree ds.

Proof. Consider the function $T(\omega) = u_1^{l_1}(\omega) \cdots u_d^{l_d}(\omega) v(\omega)$ on S^{d-1}. It can be shown by the spherical coordinate formula (11) that the functions $u_1^{l_1}(\omega) \cdots u_d^{l_d}(\omega)$ are trigonometric polynomials on the variables $\omega_1, \dots, \omega_{d-1}$ of degree at most $d-1$, and the function $v(\omega)$ is a trigonometric polynomial of degree $d' = (d-1)(d-2)/2$. Because $|l| \leqslant s$ and $s \geqslant d^2/2$ the degree of the polynomial $T(\omega)$ is $|l|(d-1) + d' \leqslant ds$. We decompose the polynomial T via its Fourier series over the variable ω_{d-1}

$$\begin{aligned} T(\omega) &= T(\omega_1, \dots, \omega_{d-1}) \\ &= B_{l0}(\omega_1, \dots \omega_{d-2}) + \sum_{m=1}^{ds} [A_{lm}(\omega_1, \dots \omega_{d-2}) \sin(m\omega_{d-1}) \\ &\quad + B_{lm}(\omega_1, \dots, \omega_{d-2}) \cos(m\omega_{d-1})], \end{aligned}$$

where the Fourier coefficients A_{lm} and B_{lm} are functions dependent only on $\omega_1, \dots \omega_{d-2}$.

Using (12) we integrate the function $T(\omega)$ over the domain $\Omega(r)$

$$\begin{aligned}\int_{\Omega(r)} T(\omega)\,d\omega = {} & \int_0^{2\pi} d\omega_1 \int_0^{\pi} d\omega_2 \cdot \ldots \cdot \\ & \times \int_0^{\pi} d\omega_{d-2} \int_0^{\arccos \alpha_r} T(\omega_1, \ldots, \omega_{d-1}) d\omega_{d-1} \\ = {} & b_{l0} \int_0^{\arccos \alpha_r} d\omega_{d-1} \\ & + \sum_{m=1}^{ds} \left[a_{lm} \int_0^{\arccos \alpha_r} \sin(m\omega_{d-1})\, d\omega_{d-1} \right. \\ & \left. + b_{lm} \int_0^{\arccos \alpha_r} \cos(m\omega_{d-1})\, d\omega_{d-1} \right], \end{aligned} \tag{14}$$

where a_{lm} and b_{lm} are coefficients dependent only on l and m.

Consider the functions

$$\cos(m \arccos t) = \sum_{k=0}^{m} p_{mk} t^k$$

and

$$\frac{\sin(m \arccos t)}{\sqrt{1-t^2}} = \sum_{k=0}^{m-1} q_{mk} t^k,$$

which are the Chebyshev polynomials of the first kind of degree m and second kind of degree $m-1$, respectively. Hence taking into consideration (14) we have

$$\begin{aligned} & \int_{\Omega(r)} T(\omega)\, d\omega \\ & = b_{l0} \arccos \alpha_r \\ & \quad + \sum_{m=1}^{ds} \frac{(-1)^{m+1}}{m} [a_{lm}(1 - \cos(\arccos \alpha_r)) + b_{lm} \sin(\arccos \alpha_r)] \\ & = b_{l0} \arccos \alpha_r \\ & \quad + \sum_{m=1}^{ds} \frac{(-1)^{m+1}}{m} \left[a_{lm} \left(1 - \sum_{k=0}^{m} p_{mk} \alpha_r^k \right) + b_{lm} \sqrt{1-\alpha_r^2} \sum_{k=0}^{m-1} q_{mk} \alpha_r^k \right]. \end{aligned}$$

Set

$$\delta_l = b_{l0}, \quad \gamma_l = b_{l0}\pi + \sum_{m=1}^{ds} \frac{(-1)^{m+1}}{m} a_{lm}$$

and

$$P_{l,s}(t) = \sum_{m=1}^{ds} \frac{(-1)^m}{m} a_{lm} \sum_{k=0}^{m} p_{mk} t^k, \quad Q_{l,s}(t) = \sum_{m=1}^{ds} \frac{(-1)^{m+1}}{m} b_{lm} \sum_{k=0}^{m-1} q_{mk} t^k.$$

Then we have

$$\int_{\Omega(r)} T(\omega)\, d\omega = \delta_l \arccos \alpha_r + \gamma_l + P_{l,s}(\alpha_r) + \sqrt{1-\alpha_r^2}\, Q_{l,s}(\alpha_r).$$

Lemma 5.2 is proved. $\square$

We continue the proof of Theorem 5.1. From (13) and Lemma 5.2 we obtain

$$I_l = \int_0^2 g(r^2) r^{|l|+d-1} \times \left[\delta_l \arccos \alpha_r + \gamma_l + P_{l,s}(\alpha_r) + \sqrt{1-\alpha_r^2}\, Q_{l,s}(\alpha_r)\right] dr. \tag{15}$$

Since $\alpha_r = r/2$ then the functions

$$P_{l,s}(\alpha_r) = \sum_{k=1}^{ds} p'_{lk} r^k \quad \text{and} \quad Q_{l,s}(\alpha_r) = \sum_{k=0}^{ds} q'_{lk} r^k$$

are polynomials in the variable r of degree ds with some coefficients p'_{lk} and q'_{lk}. Denote $\gamma'_l = \gamma_l + p'_{l0}$, and

$$u_\lambda(g) = \int_0^2 g(r^2)\, r^{\lambda+d-1} dr, \quad \lambda \in \mathbf{Z}_+$$

$$v_\lambda(g) = \int_0^2 g(r^2)\, r^{\lambda+d-1} \arccos(r/2)\, dr,$$

$$w^{(1)}_{\lambda+k}(g) = \int_0^2 g(r^2)\, r^{\lambda+k+d-1}\, dr,$$

$$w^{(2)}_{\lambda+k}(g) = \int_0^2 g(r^2)\, r^{\lambda+k+d-1} \sqrt{1-r^2/4}\, dr.$$

Using this notation we rewrite the integral (15) as follows:

$$I_l = \delta_l v_{|l|}(g) + \gamma'_l u_{|l|}(g) + \sum_{k=1}^{ds} p'_{lk} w^{(1)}_{|l|+k}(g) + \sum_{k=0}^{ds} q'_{lk} w^{(2)}_{|l|+k}(g).$$

Recall from (13) the definition $I_l \equiv \int_{B^d-e} g(|x|^2) x^l\, dx$. We substitute this expression in (10) and changing the order of summation in four summands to obtain

$$\begin{aligned}
\langle g_a, P\rangle &= \sum_{|l|\leqslant s} \pi_l(A;P) I_l \\
&= \sum_{|l|\leqslant s} \pi_l(A;P)\Bigg[\delta_l v_{|l|}(g) + \gamma_l' u_{|l|}(g) + \sum_{k=1}^{ds} p_{lk}' w_{|l|+k}^{(1)}(g) \\
&\quad + \sum_{k=0}^{ds} q_{lk}' w_{|l|+k}^{(2)}(g)\Bigg] \\
&= \sum_{m=0}^{s}\left(\sum_{|l|=m} \delta_l \pi_l(A;P)\right) u_m(g) + \sum_{m=0}^{s}\left(\sum_{|l|=m} \gamma_l' \pi_l(A;P)\right) v_m(g) \\
&\quad + \sum_{m=1}^{(d+1)s}\left(\sum_{l,k:|l|+k=m} p_{lk}' \pi_l(A;P)\right) w_m^{(1)}(g) \\
&\quad + \sum_{m=1}^{(d+1)s}\left(\sum_{l,k:|l|+k=m} q_{lk}' \pi_l(A;P)\right) w_m^{(2)}(g).
\end{aligned}$$

From here we directly obtain the statement of Theorem 5.1.

6. Partition of variables for the moments of radial functions with centers outside of S^{d-1}

Now we study the moments $\langle g_a, P_{ij}\rangle$ of the radial functions $g_a(x)$ in the general case, that is, the center a is any point in the space $\mathbf{R}^d$. In this case, we will prove the next result about the partition of the variables g and a.

Theorem 6.1. *Let $g \in C(\mathbf{R})$, $a \in \mathbf{R}^d$, $\rho = |a|$, and P be any polynomial on $\mathbf{R}^d$ of degree s, $s \geqslant d^2/2$. We separate the space $\mathbf{R}^d$ into four sets*

$$D_0 = \{0\}, \quad D_1 = \{a\colon 0 < |a| < 1\},$$

$$D_2 = \{a : |a| = 1\}, \quad D_3 = \{a : |a| > 1\}.$$

Then the following statements hold:

1. *if $v = 1$, or 3 then*

$$\langle g_a, P\rangle = \sum_{m=0}^{(2d+5)s+4} \pi_m^{(v)}(a, \rho, 1/\rho; P)\gamma_m^{(v)}(g, \rho), \quad a \in D_v,$$

where $\pi_m^{(v)}(a, \rho, 1/\rho; P)$ are polynomials in the variables a, ρ and $1/\rho$ of degree s, and $\gamma_m^{(v)}(g, \rho)$ are linear functionals in g on the space $C(\mathbf{R})$ and some functions in the variable ρ.

2. *if* $v = 0$ *or* 2 *then*

$$\langle g_a, P \rangle = \sum_{m=0}^{5s+4} \pi_m^{(v)}(a,;P)\gamma_m^{(v)}(g), \quad a \in D_v,$$

where the $\pi_m^{(v)}(a,;P)$ *are polynomials in the variable a of degree* s, *and the* $\gamma_m^{(v)}(g)$ *are linear functionals to* $C(\mathbf{R})$ *and some functions in the variable* ρ.

Proof. Let $A \in SO(d)$ be an orthogonal matrix such that $\rho Ae = a$. Then by the invariance of measure dx relative to rotations in $\mathbf{R}^d$ we have

$$\begin{aligned}\langle g_a, P \rangle &= \int_{B^d} g(|x-a|^2)P(x)\,dx = \int_{B^d} g(|x-\rho e|^2)P(Ax)\,dx \\ &= \int_{B^d-\rho e} g(|x|^2)P(A(x+\rho e))\,dx \\ &= \sum_{|l|\leqslant s} \pi_l(A,\rho;P) \int_{B^d-\rho e} g(|x|^2)x^l\,dx, \end{aligned} \tag{16}$$

where $\pi_l(A,\rho;P)$ are some polynomials of d^2+1 variables A and ρ of degree $|l|$.

We now study the integral $\int_{B^d-\rho e} g(|x|^2)x^l\,dx$ for the different values $\rho = |a|$, i.e. $a \in D_v$, $v = 0,1,2,3$.

1. *The case* $a \in D_1$: We first consider the case $0 < \rho \equiv |a| < 1$. Define in the ball $B^d - \rho e$ the surface

$$\Omega_\rho(r) = S^{d-1}(r) \bigcap (B^d - \rho e), \quad 0 \leqslant r \leqslant 1+\rho.$$

Obviously

$$\begin{aligned} B^d - \rho e &= \bigcup_{0\leqslant r\leqslant 1+\rho} \Omega_\rho(r) \\ &= \left(\bigcup_{0\leqslant r<1-\rho} S^{d-1}(r)\right) \bigcup \left(\bigcup_{1-\rho\leqslant r\leqslant 1+\rho} \Omega_\rho(r)\right). \end{aligned} \tag{17}$$

Fix r. We represent the surface $\Omega_\rho(r)$ by the spherical system of coordinates (11). In the rectangular system of coordinates the surface $\Omega_\rho(r)$ is described by the next equation and inequality

$$x_1^2 + \cdots + x_d^2 = r^2, \quad x_1^2 + \cdots + x_{d-1}^2 + (x_d - \rho)^2 \leqslant 1.$$

It follows from the spherical system of coordinates (11) that the points on the surface $\Omega_\rho(r)$ satisfy the inequality $r^2 + 2x_d\rho + \rho^2 \leqslant 1$, that is

$$r^2 + 2r\rho\cos\omega_{d-1} + \rho^2 \leqslant 1 \quad \text{or } 0 \leqslant \omega_{d-1} \leqslant \arccos\left(\frac{1-r^2-\rho^2}{-2r\rho}\right).$$

Set $\alpha_{\rho,r} = \frac{1-r^2-\rho^2}{-2r\rho}$. Then the surface $\Omega_\rho(r)$ has the form

1. If $0 \leqslant r < 1-\rho$, then $\Omega_\rho(r) = S^{d-1}(r)$, and

2. If $1-\rho \leqslant r < 1+\rho$, then

$$\Omega_\rho(r) = \{(r,\omega) : (\omega_1, \ldots, \omega_{d-1}) \in [0, 2\pi] \times [0,\pi]^{d-3} \times [0, \arccos \alpha_{\rho,r}]\}. \tag{18}$$

We fix l and consider the last integral in (16). Using the spherical coordinates (11) and (17) we represent this integral as

$$\begin{aligned}\int_{B^d-\rho e} g(|x|^2)x^l\,dx &= \int_0^{1+\rho} g(r^2)r^{|l|+d-1}\,dr \int_{\Omega_\rho(r)} u^l(\omega)v(\omega)\,d\omega \\ &= \int_0^{1-\rho} g(r^2)r^{|l|+d-1}\,dr \int_{S^{d-1}} u^l(\omega)v(\omega)\,d\omega \\ &\quad + \int_{1-\rho}^{1+\rho} g(r^2)r^{|l|+d-1}\,dr \int_{\Omega_\rho(r)} u^l(\omega)v(\omega)\,d\omega \\ &\coloneqq I_l' + I_l''. \end{aligned} \tag{19}$$

Set $b_l = \int_{S^{d-1}} u^l(\omega)v(\omega)\,d\omega$. Then

$$\begin{aligned} I_l' &= \int_0^{1-\rho} g(r^2)r^{|l|+d-1}\,dr \int_{S^{d-1}} u^l(\omega)v(\omega)\,d\omega \\ &= b_l \int_0^{1-\rho} g(r^2)r^{|l|+d-1}\,dr. \end{aligned} \tag{20}$$

Now we consider the integral

$$I_l'' = \int_{1-\rho}^{1+\rho} g(r^2)r^{|l|+d-1}\,dr \int_{\Omega_\rho(r)} u_1^{l_1}(\omega)\cdots u_d^{l_d}(\omega)v(\omega)\,d\omega. \tag{21}$$

It follows from Lemma 5.2 that

$$\begin{aligned} I_l'' = \int_0^2 g(r^2)r^{|l|+d-1}\Big[&\delta_l \arccos \alpha_{\rho,r} \\ &+ \gamma_l + P_{l,s}(\alpha_{\rho,r}) + \sqrt{1-\alpha_{\rho,r}^2}\,Q_{l,s}(\alpha_{\rho,r})\Big]\,dr. \end{aligned} \tag{22}$$

Since $\alpha_{\rho,r} = \frac{1-r^2-\rho^2}{-2r\rho}$, and the functions P_{ls}, Q_{ls} are polynomials of degree ds, then the functions

$$P_{l,s}(\alpha_{\rho,r}) = \sum_{k=-ds}^{ds} p_{lk}(\rho, 1/\rho)r^k \quad \text{and} \quad Q_{l,s}(\alpha_{\rho,r}) = \sum_{k=-ds}^{ds} q_{lk}(\rho, 1/\rho)r^k \tag{23}$$

are rational functions in the variables r of degree ds and p_{lk}, q_{lk} are some polynomials in the variables ρ, $1/\rho$ of degree ds.

Denote by $\gamma_l' = \gamma_l + p_{l0}(\rho, 1/\rho)$, and

$$h_\lambda(g,\rho) = \int_0^{1-\rho} g(r^2) r^{\lambda+d-1}\, dr, \quad \lambda \in \mathbf{Z}_+$$

$$u_\lambda(g,\rho) = \int_{1-\rho}^{1+\rho} g(r^2) r^{\lambda+d-1}\, dr,$$

$$v_\lambda(g,\rho) = \int_{1-\rho}^{1+\rho} g(r^2) r^{\lambda+d-1} \arccos(\alpha_{\rho,r})\, dr,$$

$$w_{\lambda+k}^{(1)}(g,\rho) = \int_{1-\rho}^{1+\rho} g(r^2) r^{\lambda+k+d-1}\, dr,$$

$$w_{\lambda+k}^{(2)}(g,\rho) = \int_{1-\rho}^{1+\rho} g(r^2) r^{\lambda+k+d-1} \sqrt{1-\alpha_{\rho,r}^2/4}\, dr.$$

We rewrite the integral (20)–(23) as follows: $I_l' = b_l h_{|l|}(g,\rho)$ and

$$I_l'' = \delta_l v_{|l|}(g,\rho) + \gamma_l' u_{|l|}(g,\rho) + \sum_{k=-ds}^{ds} p_{lk}(\rho, 1/\rho) w_{|l|+k}^{(1)}(g,\rho) + \sum_{k=-ds}^{ds} q_{lk}(\rho, 1/\rho) w_{|l|+k}^{(2)}(g,\rho).$$

We substitute this expression in (16) to obtain:

$$\begin{aligned} \langle g_a, P \rangle &= \sum_{|l| \leqslant s} \pi_l(A, \rho; P)(I_l' + I_l'') \\ &= \sum_{|l| \leqslant s} \pi_l(A, \rho; P)[b_l h_{|l|}(\rho, 1/\rho) + \delta_l v_{|l|}(g,\rho) + \gamma_l' u_{|l|}(g,\rho)] \\ &\quad + \sum_{|l| \leqslant s} \pi_l(A, \rho; P) \Bigg[\sum_{k=-ds}^{ds} p_{lk}(\rho, 1/\rho) w_{|l|+k}^{(1)}(g,\rho) \\ &\quad + \sum_{k=-ds}^{ds} q_{lk}(\rho, 1/\rho) w_{|l|+k}^{(2)}(g,\rho) \Bigg]. \end{aligned}$$

Hence changing the order of summation we obtain

$$\begin{aligned}\langle g_a, P\rangle = &\sum_{m=0}^{s}\left(\sum_{|l|=m} b_l\pi_l(A,\rho;P)\right)h_m(g,\rho)\\ &+\sum_{m=0}^{s}\left(\sum_{|l|=m} \delta_l\pi_l(A,\rho;P)\right)v_m(g,\rho)\\ &+\sum_{m=0}^{s}\left(\sum_{|l|=m} \gamma'_l\pi_l(A,\rho;P)\right)u_m(g,\rho)\\ &+\sum_{m=0}^{s}\left(\sum_{l,k:|l|+k=m} p_{lk}(\rho,1/\rho)\pi_l(A,\rho;P)\right)w_m^{(1)}(g)\\ &+\sum_{m=0}^{s}\left(\sum_{l,k:|l|+k=m} p_{lk}(\rho,1/\rho)\pi_l(A,\rho;P)\right)w_m^{(2)})(g).\end{aligned}$$

From here we directly obtain the statement of Theorem 6.1.

2. *Case* $a=0$: From (16) we have

$$\langle g_0, P\rangle = \int_{B^d} g(|x|^2)P(x)\,dx = \sum_{|l|\leqslant s}\pi_l(0,0;P)\int_{B^d} g(|x|^2)x^l\,dx. \tag{24}$$

According to (20) we obtain

$$\int_{B^d} g(|x|^2)x^l\,dx = b_l\int_0^1 g(r^2)r^{|l|+d-1}\,dr. \tag{25}$$

Set

$$\pi_m^{(0)}(P) = \sum_{|l|=m} b_l\pi_l(0,0;P) \quad \text{and} \quad \gamma_m^{(0)}(g) = \int_{B^d} g(r^2)r^{m+d-1}\,dr.$$

Then from (24) and (25) it follows that

$$\begin{aligned}\langle g_0, P\rangle &= \sum_{m=0}^{s}\left(\sum_{|l|=m} b_l\pi_l(0,0;P)\right)\int_{B^d} g(r)r^{m+d-1}\,dr\\ &= \sum_{m=0}^{s}\pi_m^{(0)}(P)\gamma_m^{(0)}(g).\end{aligned}$$

3. *Case* $|a|=1$: This case was proved in Section 5.

4. *Case* $|a|>1$: In analogy to (16) and (19), we have

$$\langle g_a, p\rangle = \sum_{|l|\leqslant s} \pi_l(A,\rho;P)\int_{\rho-1}^{\rho+1} g(r)r^{|l|+d-1}\,dr \times \int_{S^{d-1}(r)\cap(B^d-\rho e)} u^l(\omega)v(\omega)\,d\omega.$$

We now repeat the proof of the case $0<|a|<1$. Theorem 6.1 is proved. □

We formulate one consequence of Theorem 6.1 in a form suitable for future use.

Consequence 6.2. *Under the conditions of Theorem* 6.1 *for every* $v=0,1,2,3$ *the following relation is true*

$$\langle g_a, P\rangle = \sum_{m=0}^{5ds} \pi_m^{(v)}(A,\rho,1/\rho;P)\gamma_m^{(v)}(g,\rho),\quad a\in D_v,$$

where $\pi_m^{(v)}(a,\rho,1/\rho;P)$ *are polynomials in the variables* A, ρ *and* $1/\rho$ *of degree* s, *and* $\gamma_m^{(v)}(g,\rho)$ *are linear functionals of* $g\in C(\mathbf{R})$ *and some functions of* ρ.

7. When the polynomial space belongs to the manifold of radial functions

In this section, we will show that if n and s are chosen in a special way, then the polynomial space $\mathscr{P}_{ds}$ belongs to the manifold of the radial functions $\mathscr{R}(a_1,\dots,a_n)$ with some centers $a_1,\dots,a_n$ belonging to the unit sphere.

Lemma 7.1. *Let* n *and* s *be any natural numbers such that* $n>c\dim\mathscr{P}_{ds}^{\mathrm{hom}}$, *where* $c>1$ *is some constant dependent only on* d. *Then there exist points* $a_1,\dots,a_n$ *on the unit sphere* S^{d-1} *such that*

$$\mathscr{P}_{d,s}\subset\mathscr{R}(a_1,\dots,a_n).$$

Proof. We will find the points $a_1\dots,a_n\in S^{d-1}$ satisfying the following conditions. For any polynomial P from the space $\mathscr{P}_{d,s}$ there exist polynomials $g_1,\dots,g_n\in\mathscr{P}_{1,s/2}$ of degree $s/2$ such that

$$P(x)=\sum_{k=1}^{n} g_k(|x-a_k|^2). \tag{26}$$

Let Q be any polynomial from the space $\mathscr{P}_{d,s}$. Consider the orthonormal system of polynomials $\Pi=\{P_{ij}\}_{(i,j)\in I}$ introduced in Section 3. Let s be any natural number. We examine in the set I the subset of the matching indices $I_s=\{(i,j)\in I: j\leqslant s\}$, and consider in the function system Π the finite subsystem $\Pi_s=\{P_{ij}\}_{(i,j)\in I_s}$. From Consequence 3.2 it follows that the set Π_s is an orthonormal basis in the space of polynomials $\mathscr{P}_{d,s}$. Decompose the polynomial Q by the system Π_s

$$Q(x) = \sum_{(i,j)\in I_s} b_{ij}(Q)P_{ij}(x), \quad b_{ij} = \langle Q, P_{ij} \rangle. \tag{27}$$

We now investigate the decomposition of an arbitrary radial function using the orthogonal system Π. Let a be any point on the sphere $\in S^{d-1}$ and g be any polynomial. Consider the radial function $g_a(x) = g(|x-a|^2)$ with center a. Let $(i,j)\in I$ be a fixed matching index and $b_{ij}(g_a) = \langle g_a, P_{ij} \rangle$ be the corresponding moments of the function g_a by the system Π. By Lemma 4.2 we have

$$b_{ij}(g_a) = \sum_{i'=0}^{\infty} t_{ii'}(A)b_{i'j}(g_e), \tag{28}$$

where $A\in SO(d)$ is an orthogonal matrix such that $Ae = a$, $e = (0, \dots, 0, 1)$. It follows from Theorem 5.1 that

$$b_{ij}(g_e) = \sum_{m=0}^{vj} \pi_m(e; P_{ij})\gamma_m(g), \tag{29}$$

where the $\gamma_m(g)$ are some linear functionals on $C(\Delta)$, and $v = 2d+5$. Put $\pi_{mij} = \pi_m(e; P_{ij})$. Due to relations (28) and (29) we have

$$b_{ij}(g_a) = \sum_{m=0}^{vj} \left(\sum_{i'=0}^{\infty} t_{ii'}(A)\pi_{mi'j} \right) \gamma_m(g). \tag{30}$$

Now consider an arbitrary linear combination of n radial functions

$$G(x) = \sum_{k=1}^{n} g_k(|x-a_k|^2), \quad a_k\in S^{d-1}, \quad g_k\in\mathscr{P}_{1,s/2}.$$

Let the matrices $A_k\in SO(d)$ be such that $A_k e = a_k$. Then by (30) the moments of the function G are equal to

$$b_{ij}(G) = \sum_{k=1}^{n} \sum_{m=0}^{vj} \left(\sum_{i'=0}^{\infty} t_{ii'}(A_k)\pi_{mi'j} \right) \gamma_m(g_k). \tag{31}$$

Consider two sets of the matching indices $I_s = \{(i,j)\in I : j \leqslant s\}$ and

$$K_s = \{(k,m) : k = 1, \dots, n;\ m = 0, 1, \dots, vs\}.$$

Let $\mathscr{A} = (A_1, \dots, A_n)$ be a vector with matrix coordinates $A_1, \dots, A_n$ belonging to the orthogonal group $SO(d)$. For every matching index $(i,j)\in I_s$ and $(k,m)\in K_s$ we denote

$$Z_{ij}^{km}(\mathscr{A}) = \begin{cases} \sum_{i'=0}^{\infty} t_{ii'}(A_k)\pi_{mi'j}, & 0 \leqslant m \leqslant vj \\ 0, & vj < m \leqslant vs. \end{cases} \tag{32}$$

Construct the matrix

$$Z(\mathscr{A}) = (Z_{ij}^{km}(\mathscr{A}))_{(i,j)\in I_s}^{(k,m)\in K_s},$$

where (k, m) is the indicator number of the column of the matrix $Z(\mathscr{A})$, and (i,j) is the indicator number of the row of the matrix $Z(\mathscr{A})$. The order of the matrix $Z(\mathscr{A})$ is equal to $|I_s| \times |K_s|$. We denote $\gamma_{km} = \gamma_m(g_k)$ and construct the vectors

$$\gamma = (\gamma_{km})_{(k,m) \in K_s}, \quad b = (b_{ij}(G))_{(i,j) \in I_s},$$

where the coordinates are numbered by the matching indices $(k, m) \in K_s$ and $(i,j) \in I_s$, respectively. With the help of the matrix $Z(\mathscr{A})$ and the vectors b and γ expression (31) may be rewritten as

$$\sum_{(k,m) \in K_s} Z_{ij}^{km}(\mathscr{A})\gamma_{km} = b_{ij}(G), \quad (i,j) \in I_s. \tag{33}$$

To show that the given polynomial Q belongs to the manifold $\mathscr{R}(a_1, \dots, a_n)$ one needs to prove that for some selection of function $G \in \mathscr{R}(a_1, \dots, a_n)$ all moments of the functions Q and G coincide, that is

$$b_{ij}(G) = b_{ij}(Q), \quad (i,j) \in I_s. \tag{34}$$

Construct the vector $\hat{Q} = (b_{ij}(\hat{Q}))_{(i,j) \in I_s}$. Using (33) relation (31) may be rewritten in the matrix form

$$Z(\mathscr{A})\gamma = \hat{Q}. \tag{35}$$

Thus, we obtain a finite-dimensional linear system of equations relative to the unknown γ.

We need to show that for some choice of the matrix set $\mathscr{A} = (A_1, \dots, A_n)$ system (35) has a solution.

According to the Kronecker–Capelli Theorem it is sufficient to show that there exist a collection of matrices $\mathscr{A}^* = (A_1^*, \dots, A_n^*)$ in the group $SO(d)$ such that the ranks of matrix $Z(\mathscr{A}^*)$ and the extended matrix $(Z(\mathscr{A}^*), \hat{Q})$ coincide. Recall that $Q \in \mathscr{P}_{d,s}$ and $dim \mathscr{P}_{d,s} = |I_s|$, where we denote $|I_s| = card\, I_s$. Therefore, it is enough to prove that the rank of the matrix $Z(\mathscr{A}^*)$ for some collection of matrices $\mathscr{A}^*$ is equal

$$rank\, Z(\mathscr{A}^*) = |I_s|.$$

Fix an index j' in the set $\{1, \dots, s\}$. Consider the submatrix of matrix $Z(\mathscr{A})$

$$Z(\mathscr{A}, j') = (Z_{ij'}^{k,vj'}(\mathscr{A}))_{(i,j') \in I_{j'}}^{(k,j') \in I_{j'}},$$

where k and i are the indices of the columns and rows of the matrix $Z(\mathscr{A}, j')$ of order $|I_{j'} \diagdown I_{j'-1}| \times n$. Then the matrix $Z(\mathscr{A})$ has the form

$$Z(\mathscr{A}) = \begin{pmatrix} Z(\mathscr{A}, 1) & 0 & \dots & 0 \\ B & Z(\mathscr{A}, 2) & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ C & D & \dots & Z(\mathscr{A}, s) \end{pmatrix},$$

where $B, C, D, \dots$ are some nonzero matrices.

Recall that $|I_{j'}| = dim\, \mathscr{P}_{d,j'}$, and $|I_{j'} \setminus I_{j'-1}| = dim\, \mathscr{P}^{\text{hom}}_{d,j'}$. Since $Z^{km}_{ij}(\mathscr{A}) = 0$ for any $m > vj$ (see (32)) then

$$rank\, Z(\mathscr{A}) \geqslant \sum_{j=0}^{s} rank\, Z(\mathscr{A}, j). \tag{36}$$

Moreover, we note that as a consequence of the inequality $dim\, \mathscr{P}^{\text{hom}}_{d,s} \geqslant dim\, \mathscr{P}^{\text{hom}}_{d,j} = |I_j \setminus I_{j-1}|$ the number of rows $|I_j \setminus I_{j-1}|$ of the matrix $Z(\mathscr{A}, j)$ is at most the number of columns n. For fixed index j' we calculate the rank of the matrix $Z(\mathscr{A}, j')$. The elements of the matrix $Z(\mathscr{A}, j)$ are

$$Z^{k,vj}_{ij}(\mathscr{A}) = \sum_{i'=0}^{\infty} t_{ii'}(A_k)\pi_{vj,i',j}.$$

According to Property 2 in Section 4 of the matrix $T(A)$, the family $\{t_{ii'}(A)\}$ $i, i' = 0, 1, \ldots,$ is a linear independent system of functions on the matrix set $SO(d)$. Therefore, taking into account $\pi_{vj,i',j} = \pi_{vj}(e; P_{i'j}) \neq 0$ for every i' the family of functions

$$\left\{ \phi_{ij}(A) := \sum_{i'=0}^{\infty} t_{ii'}(A)\pi_{vj,i',j}, \quad (i,j) \in I_s \right\}$$

is also a linear independent system of functions on $SO(d)$. It follows that there exist matrices $A^*_1, \ldots, A^*_n \in SO(d)$ such that for any $j \in \{1, \ldots, s\}$ the rank of the matrix

$$Z(\mathscr{A}^*, j) = (Z^{k,vj}_{ij}(\mathscr{A}^*))^{(k,j) \in I_j}_{(i,j) \in I_j}, \quad \mathscr{A}^* = (A^*_1, \ldots, A^*_n)$$

is equal to the number of rows of the matrix $Z(\mathscr{A}^*, j)$, that is

$$rank\, Z(\mathscr{A}^*, j) = |I_j \setminus I_{j-1}|.$$

Since $1 \leqslant j \leqslant s$, and $I_{-1} = \emptyset$, then from (36) we obtain

$$rank\, Z(\mathscr{A}^*) \geqslant \sum_{j=0}^{s} rank\, Z(\mathscr{A}^*, j) \geqslant \sum_{j=0}^{s} |I_j \setminus I_{j-1}| = |I_s|.$$

Since the number of rows of the matrix $Z(\mathscr{A}^*)$ is equal $|I_s|$ then we obtain that $rank\, Z(\mathscr{A}^*) = |I_s|$. Lemma 7.1 is proved. $\square$

As we show next, the upper bound in Theorem 2.1 is now a simple consequence of Lemma 7.1.

Proof of Theorem 2.1 (Upper bound). We resort to the Jackson Theorem [47] confirming that the distance of Sobolev class $W^{r,d}_2$ from the space of polynomials $\mathscr{P}_{d,s}$ of degree s satisfies the inequality

$$dist\,(W^{r,d}_2, \mathscr{P}_{d,s}, L_2) \leqslant cs^{-r},$$

where the constant c depends only on r and d.

Let the natural number s and n be chosen such that $c\,dim\,\mathscr{P}_{d,s}^{\text{hom}} < n \leqslant 2c\,dim\,\mathscr{P}_{d,s}^{\text{hom}}$ where $c > 1$ is some absolute constant. Since $dim\,\mathscr{P}_{d,s}^{\text{hom}} \asymp s^{d-1}$ (see [43]), it follows that $n \asymp s^{d-1}$. Then by Lemma 7.1, $\mathscr{P}_{d,s}$ belongs to the manifold $\mathscr{R}(a_1, \dots, a_n)$ for some points $a_1, \dots, a_n \in S^{d-1}$. Hence we obtain

$$dist\,(W_2^{r,d}, \mathscr{R}(a_1, \dots, a_n), L_2) \leqslant dist\,(W_2^{r,d}, \mathscr{P}_{d,s}, L_2) \leqslant cs^{-r} \asymp n^{-\frac{r}{d-1}}.$$

The upper bound in Theorem 2.1 is proved. $\square$

8. Approximation of polynomials of high degree by radial functions

Introduce in the space $\mathscr{P}_{d,s}$ of polynomials of degree s on B^d the norm from the space $L_2(B^d)$. Let

$$B\mathscr{P}_{d,s} = \{P \in \mathscr{P}_{d,s} : ||P||_{L_2(B^d)} \leqslant 1\}$$

be the unit ball in the space $\mathscr{P}_{d,s}$. In Section 7, we proved that if the natural numbers s and n are such that $n > c\,dim\,\mathscr{P}_{d,s}^{\text{hom}}$ then the polynomial space $\mathscr{P}_{d,s} \subset \mathscr{R}(a_1, \dots, a_n)$. We will now show that for $n < c'\,dim\,\mathscr{P}_{d,s}^{\text{hom}}$ the space $\mathscr{P}_{d,s}$ dows not belong to $\mathscr{R}_n$, where $0 < c' < 1$ is some absolute constant. Moreover the space $\mathscr{P}_{d,s}$ is "badly" approximated by the manifold $\mathscr{R}_n$. That is, the deviation of the function class $B\mathscr{P}_{d,s}$ from $\mathscr{R}_n$ satisfies the inequality

$$dist\,(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2)) \geqslant c_1 > 0,$$

where c_1 depends only on d. The proof of this statement is based on a scheme proposed by Maiorov [20]. The main idea of the proof is the comparison of the entropy numbers of the sets $B\mathscr{P}_{d,s}$ and $B\mathscr{R}_n$, where we denote by $BQ = \{f \in Q : ||f||_{L_2} \leqslant 1\}$ the intersection of the set of functions Q and the unit ball in the space L_2.

Let $\Pi = \{P_{ij}\}_{(i,j) \in I}$ be the orthonormal system of polynomials introduced in Section 3. We fix a natural number s. Consider in the index set I the subset $I_s = \{(i,j) \in I : j \leqslant s\}$ and the corresponding subset in Π of polynomials

$$\Pi_s = \{P_{ij}\}_{(i,j) \in I_s}.$$

Set $m = m_s = card\,I_s$. We have $m_s \asymp s^d$. Arrange the set I_s such that $I_s = \{1, \dots, m\}$ and correspondingly the set of polynomials $\Pi_s = \{P_k\}_{k=1}^m$.

By Consequence 3.2, we have that the system of polynomials Π_s is an orthonormal basis in the space $\mathscr{P}_{d,s}$. Hence every polynomial $P \in \mathscr{P}_{d,s}$ may be represented as

$$P(x) = \sum_{k=1}^{m} b_k(P) P_k(x), \quad b_k(P) = \langle P, P_k \rangle,$$

where the $b_k(P)$ are the moments of the function P relative to the basis Π.

Consider the normed space l_2^m consisting of the complex vectors $b = (b_1, \dots, b_m)$ with norm

$$||b||_2 = \left(\sum_{k=1}^{m} |b_k|^2 \right)^{1/2}.$$

Denote by $B_2^m = \{b \in l_2^m : ||b||_2 \leqslant 1\}$ the unit ball in the space l_2^m. By Parseval's equality $||P||_{L_2}^2 = \sum_{k=1}^{m} |b_k(P)|^2$. Therefore, the polynomial ball $B\mathscr{P}_{d,s}$ is isometric to the unit ball B_2^m in the space l_2^m.

Introduce in the space l_2^m the set of sign-valued vectors

$$E_m = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) : \varepsilon_1, \dots, \varepsilon_m = \pm 1\}.$$

Note that the set of vectors $m^{-1/2}E^m$ belongs to the ball B_2^m.

Let F be some set of functions in the space L_2. We denote by $\hat{F}$ the corresponding set $\hat{F} = \{(b_1(f), \dots, b_m(f)) : f \in F\}$ of vectors in the space l_2^m.

From the above and Bessel's inequality it follows that the deviation of the ball $B\mathscr{P}_{d,s}$ from the manifold $\mathscr{R}_n$ satisfies the inequality

$$\begin{aligned} dist\,(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2)) &\geqslant dist\,(\widehat{B\mathscr{P}}_{d,s}, \hat{\mathscr{R}}_n, l_2^m) \\ &\geqslant m^{-1/2}\, dist\,(E^m, \hat{\mathscr{R}}_n, l_2^m). \end{aligned} \tag{37}$$

Consider the function: $sgn\,x = 1$ if $x \geqslant 0$, and $sgn\,x = -1$ if $x < 0$. If $a = (a_1, \dots, a_m) \in \mathbf{R}^m$ then we denote $sgn\,a = (sgn\,a_1, \dots, sgn\,a_m)$. If A is some set in $\mathbf{R}^m$ then denote by $sgn\,A = \{sgn\,a : a \in A\}$.

Let q be any complex number, and $\operatorname{Re} q$ be the real part of q. For any number $\alpha = \pm 1$ we have $|\alpha - q| \geqslant |\alpha - \operatorname{Re} q| \geqslant \frac{1}{2}|\alpha - sgn\,(\operatorname{Re} q)|$. Therefore for any vectors $\varepsilon \in E^m$ and $b \in \hat{\mathscr{R}}_n$ we have

$$||\varepsilon - b||_2 \geqslant \frac{1}{2}||\varepsilon - sgn(\operatorname{Re} b)||_2,$$

where $sgn(\operatorname{Re} b) = (sgn(\operatorname{Re} b_1), \dots, sgn(\operatorname{Re} b_m))$. Hence it follows that

$$\begin{aligned} dist\,(E^m, \hat{\mathscr{R}}_n, l_2^m) &= \max_{\varepsilon \in E^m} \inf_{b \in \hat{\mathscr{R}}_n} ||\varepsilon - b||_2 \\ &\geqslant \tfrac{1}{2} \max_{\varepsilon \in E^m} \min_{b \in \hat{\mathscr{R}}_n} ||\varepsilon - sgn(\operatorname{Re} b)||_2. \end{aligned} \tag{38}$$

For any positive number δ we introduce the δ-packing number of the set E^m:

$$\begin{aligned} N_\delta = \max\{N :\ &\text{there exist } \varepsilon^1, \dots, \varepsilon^N \in E^m \text{ s.t. } ||\varepsilon^i - \varepsilon^j||_2 \\ &\geqslant \delta \text{ for any } i \neq j\}. \end{aligned} \tag{39}$$

Let $\mathscr{H}(\delta) = \{\varepsilon^1, \dots, \varepsilon^{N_\delta}\}$ be the subset of elements in E^m for which the maximum in (39) is attained, that is $||\varepsilon^i - \varepsilon^j||_2 \geqslant \delta$ for any $i \neq j$. Set $\delta = \sqrt{m}/2$. It is known ([19], see also [25]) that the cardinality of the set $\mathscr{H}(\sqrt{m}/2)$ satisfies

$$card\,\mathscr{H}(\sqrt{m}/2) \geqslant 2^{c_0 m}, \tag{40}$$

where $0 < c_0 < 1$ is some absolute constant. Consider the subset of vectors in E^m

$$V_n^m := V_n^{m_s} := \{sgn(\operatorname{Re} b) : b \in \hat{\mathscr{R}}_n\}.$$

From inequalities (37) and (38) the next result follows.

Lemma 8.1. *Let s and n be any natural numbers, $m_s = card\, I_s$, and $\delta_s = \sqrt{m_s}/2$. Then in the set E^{m_s} there exist a subset $\mathscr{H}(\delta_s)$ such that*

$$dist\,(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2)) \geqslant \frac{1}{2\sqrt{m_s}} dist(\mathscr{H}(\delta_s), V_n^{m_s}, l_2^{m_s}),$$

and $card\, \mathscr{H}(\delta_s) \geqslant 2^{c_0 m_s}$.

Furthermore, we show the following estimate for the cardinality of the set $V_n^{m_s}$.

Lemma 8.2. *There exist some absolute constants $0 < c' < 1$ and $0 < c_1 < c_0/2$ such that if s and n are any natural numbers satisfying $n < c'\, dim\, \mathscr{P}_{d,s}^{\mathrm{hom}}$ and $m_s = card\, I_s$ then the inequality*

$$card\, V_n^{m_s} \leqslant 2^{c_1 m_s},$$

holds.

We will prove Lemma 8.2 in the next section. From Lemmas 8.1 and 8.2 we have the following theorem:

Theorem 8.3. *Let s and n be any natural numbers such that $n < c' \mathscr{P}_{d,s}^{\mathrm{hom}}$, where $0 < c' < 1$ is some constant depending only on d. Then*

$$dist\,(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2)) \geqslant \tfrac{1}{8}.$$

Proof. Consider the set $\mathscr{H}(\delta_s)$, $\delta_s = \sqrt{m_s}/2$. By (40) we have $card\, \mathscr{H}(\delta_s) \geqslant 2^{c_0 m_s}$. For any $\varepsilon' \neq \varepsilon'' \in \mathscr{H}(\delta_s)$ we have the inequality $||\varepsilon' - \varepsilon''||_2 \geqslant \delta_s$.

By Lemma 8.2 the cardinality $card\, V_n^{m_s} \leqslant 2^{(c_0/2)m_s}$. Therefore, there exists an element $\varepsilon^* \in \mathscr{H}(\delta_s)$ satisfying

$$\min_{v \in V_n^{m_s}} ||\varepsilon^* - v||_2 \geqslant \frac{1}{4}\sqrt{m_s}.$$

Hence, taking into consideration Lemma 8.1 we obtain

$$dist\,(B\mathscr{P}_{d,s}, \mathscr{R}_n, L_2)) \geqslant \frac{1}{2\sqrt{m_s}} \min_{v \in V_n^{m_s}} ||\varepsilon^* - v||_2 \geqslant 1/8.$$

Theorem 8.3 is proved. □

Note that from Theorem 8.3 and Lemma 7.1 Consequence 2.3 directly follows.

Let I_s^0 be some subset of the set I_s. Consider the subspace in $\mathscr{P}_{d,s}$ defined as

$$Q_{d,s}(I_s^0) = \left\{ \sum_{(i,j)\in I_s^0} \varepsilon_{ij} P_{ij}(x) : \varepsilon_{ij} = \pm 1 \textit{ for all } (i,j)\in I_s^0 \right\}.$$

From the proof of Theorem 8.3 we directly obtain.

Consequence 8.4. *Let $0<c\leqslant 1$ be any constant. If card $I_s^0 \geqslant c$ card I_s then*

$$dist\,(Q_{d,s}(I_s^0), \mathscr{R}_n, L_2) \geqslant c_1 > 0,$$

where c_1 is some constant dependent only on d.

Proof of Theorem 2.1 (Lower Bound). It suffices to prove Theorem 2.1 for natural numbers s and n satisfying to $s^{d-1} = n$, where s is an even integer. Let r be any positive number, and r' be the smallest even number such that $r' \geqslant r$. Set $\mu := \mu_r := 2r' - 1$. Let j be any number from the set $\{0, 2, \ldots, s\}$. Denote by α_j and β_j integers such that $s - j = (\mu+1)\alpha_j + \beta_j$, where $\alpha_j \in \mathbf{Z}$, and $\beta_j \in \{0, \ldots, \mu\}$.

Introduce the even index set $I_s^{\text{even}} = \{(i,j)\in I_s : j \in \{0, 2, \ldots, s\}\}$. Consider the function a from I_s^{even} to $\mathbf{R}$ defined by

$$a(i,j) = a_{ij} = (-1)^{\mu-\beta_j} \binom{\mu}{\beta_j} \gamma_j \varepsilon_{i\alpha_j},$$

where $\varepsilon_{i\alpha_j}$ is some number equal to -1 or 1. The set of functions a corresponding to all possible selections of the $\varepsilon_{i\alpha_j} = \pm 1$, $(i,j)\in I_s^{\text{even}}$ will be denoted A_s^r.

We consider in the set I_s^{even} the subset $I_{s,0}^{\text{even}} = \{(i,j)\in I_s^{\text{even}} : \beta_j = 0\}$. Construct two sets of polynomials on $\mathbf{R}^d$

$$Q(A_s^r) = \left\{ \sum_{(i,j)\in I_s^{\text{even}}} a_{ij} P_{ij}(x):\ a \in A_s^r \right\},\quad Q(A_s^r, I_{s,0}^{\text{even}}) = \left\{ \sum_{(i,j)\in I_{s,0}^{\text{even}}} a_{ij} P_{ij}(x):\ a \in A_s^r \right\}.$$

In the work of Maiorov [20] it was proved that $s^{-r} Q(A_s^r) \in c W_2^{r,d}$ with some positive constant c.

However, we have *card* $I_{s,0}^{\text{even}} \geqslant (1/\mu)$ *card* I_s^{even}. Therefore by Consequence 8.4 and using the Bessel's inequality we obtain

$$\begin{aligned} &dist\,(W_2^{r,d}, \mathscr{R}_n, L_2) \\ &\quad \geqslant c^{-1} s^{-r}\, dist(Q(A_s^r), \mathscr{R}_n, L_2) \geqslant c^{-1} s^{-r}\, dist(Q(A_s^r, I_{s,0}^{\text{even}}), \mathscr{R}_n, L_2) \\ &\quad = c^{-1} s^{-r}\, dist(Q_{d,s}(I_s^0), \mathscr{R}_n, L_2) \geqslant c^{-1} c_1 s^{-r} = c_2 n^{-r/(d-1)}. \end{aligned}$$

Theorem 2.1 is proved. $\square$

9. The proof of Lemma 8.2

We prove Lemma 8.2 according to the scheme proposed by Maiorov [20] while formulating some of those results.

Let m, s, p and q be some natural numbers. Let $\pi_{\alpha\beta}(\sigma)$, $\alpha = 1, \dots, m$; $\beta = 1, \dots, q$ be any algebraic polynomials with real coefficients of degree s in the variables $\sigma = (\sigma_1, \dots \sigma_p) \in \mathbf{R}^p$. Construct polynomials on the $p+q$ variables $\gamma = (\gamma_1, \dots, \gamma_q) \in \mathbf{R}^q$ and $\sigma = (\sigma_1, \dots \sigma_p) \in \mathbf{R}^p$:

$$\pi_\alpha(\gamma, \sigma) = \sum_{\beta=1}^{q} \gamma_\beta \pi_{\alpha\beta}(\sigma), \quad \alpha = 1, \dots, m$$

Consider in the space $\mathbf{R}^m$ the real algebraic manifold

$$\Pi_{m,s.p,q} = \{\pi(\gamma, \sigma) = (\pi_1(\gamma, \sigma), \dots \pi_m(\gamma, \sigma)) : (\gamma, \sigma) \in \mathbf{R}^q \times \mathbf{R}^p\}.$$

Introduce the set of sgn-values vectors

$$sgn\, \Pi_{m,s.p,q} = \{sgn\, \pi(\gamma, \sigma) : (\gamma, \sigma) \in \mathbf{R}^q \times \mathbf{R}^p\}.$$

Lemma 9.1 (see Maiorov [20]). *Let m, s, p, q be integers such that $p + q \leqslant m/2$. Then for the cardinality of the set $sgn(\Pi_{m,s,p,q})$ the following estimate holds*:

$$card\{sgn(\Pi_{m,s,p,q})\} \leqslant (4s)^p (p+q+1)^{p+2} \left(\frac{2em}{p+q}\right)^{p+q}.$$

Let $m = m_s$, and we consider the set of sign-vectors introduced in Section 8:

$$V_n^{m_s} = \{(sgn(Re\, b_1), \dots, sgn(Re\, b_m)) : (b_1, \dots, b_m) \in \mathscr{R}_n\}.$$

We have $b = (b_1, \dots, b_m) = (b_1(g), \dots, b_m(g))$ for some function $g \in \mathscr{R}_n$ (see Section 8). We represent the vector b as

$$b = (b_{ij}(g))_{(i,j) \in I_s} = (\langle g, P_{ij} \rangle)_{(i,j) \in I_s}.$$

Let $g_a(x) = g_a(|x-a|^2)$ be any radial function from the manifold $\mathscr{R}_n$ with center $a \in \mathbf{R}^d$. According to Consequence 6.2 for any matching index $(i,j) \in I_s$ the corresponding moments $b_{ij}(g_a) = \langle g_a, P_{ij} \rangle$ are represented by

$$\langle g_a, P_{ij} \rangle = \sum_{l=0}^{5ds} \gamma_l^{(v)}(g, \rho) \pi_l^{(v)}(a, \rho, 1/\rho; P_{ij}), \quad a \in D^v, \quad \rho = |a|, \tag{41}$$

where $v = 0, 1, 2, 3$; for every v the functions $\gamma_l^{(v)}(g, \rho)$ are linear functionals relative to $g \in C(\mathbf{R})$ and some functions of the variable ρ, and the functions $\pi_l^{(v)}(a, \rho, 1/\rho; P_{ij})$ are polynomials of degree s in the variables a, ρ and $1/\rho$.

Now let $G_{g,a} \in \mathcal{R}_n$ be any function, i.e $G_{g,a}$ be the linear combination of n radial functions with centers $a_1, \dots, a_n \in \mathbf{R}^d$

$$G_{g,a}(x) = \sum_{k=1}^{n} g_k(|x - a_k|^2), \quad g = \{g_k\}_{k=1}^n, \quad a = \{a_k\}_{k=1}^n.$$

By (41), for any index $(i,j) \in I_s$ the moments of the function $G_{g,a}$ are equal to

$$\langle G_{g,a}, P_{ij} \rangle = \sum_{k=1}^{n} \sum_{l=0}^{5ds} \gamma_l^{(v_k)}(g_k, \rho_k) \pi_l^{(v_k)}(a_k, \rho_k, 1/\rho_k; P_{ij}),$$
$$a_k \in D^{v_k}, \quad \rho_k = |a_k|,$$

where the indices $v_k = 0, 1, 2, 3$ are chosen according to the vector a_k belonging to the domain D^{v_k}. Thus the vector of moments of the function $G_{g,a}$ have the coordinates

$$\begin{aligned} b(G_{g,a}) &:= (\langle G_{g,a}, P_{ij} \rangle)_{(i,j) \in I_s} \\ &= \left(\sum_{k=1}^{n} \sum_{l=0}^{5ds} \gamma_l^{(v_k)}(g_k, \rho_k) \pi_l^{(v_k)}(a_k, \rho_k, 1/\rho_k; P_{ij} \right)_{(i,j) \in I_s}. \end{aligned} \tag{42}$$

Consider in the space l_2^m the set $b(\mathcal{R}^n) = \{b(G_{g,a}) : G_{g,a} \in \mathcal{R}^n\}$. We estimate the cardinality of the set

$$sgn\,(\mathrm{Re}\, b(\mathcal{R}_n)) = \{sgn(\mathrm{Re}\, b(G_{g,a})) : G_{g,a} \in \mathcal{R}_n\}.$$

We have

$$\begin{aligned} &card\{sgn(\mathrm{Re}\, b(\mathcal{R}_n))\} \\ &\quad \leqslant card\Bigg\{ \bigcup_{v_1,\dots,v_n=0,1,2,3} \{sgn(\mathrm{Re}\, b(G_{g,a})) : \\ &\qquad a_k \in D^{v_k},\ g_k \in C(\mathbf{R}),\ k = 1,..,n\}\Bigg\} \\ &\quad \leqslant 4^n \max_{v_1,\dots,v_n=0,1,2,3} card\{sgn(\mathrm{Re}\, b(G_{g,a})) : \\ &\qquad a_k \in D^{v_k},\ g_k \in C(\mathbf{R}),\ k = 1, \dots, n\}. \end{aligned} \tag{43}$$

Fix the indices $v_1, \dots, v_n$. We order the matching indices (i,j), (k,l) and also the functions γ, π as follows:

1. Enumerate the set $\{(i,j) \in I_s\}$ in $\{\alpha = 1, \dots, m\}$, where $m = card\, I_s$.
2. Enumerate the set $K_s \equiv \{(k,l) : k = 1, \dots, n,\ l = 0, \dots, 5ds\}$ in $\{\beta = 1, \dots, q\}$, where $q = n(5ds + 1)$.
3. Arrange in order the coordinates of the vectors $a_1, \dots, a_n \in \mathbf{R}^d$ and numbers $\rho_1, \dots, \rho_n, \frac{1}{\rho_1}, \dots, \frac{1}{\rho_n}$ in one vector $\sigma = (\sigma_1, \dots, \sigma_p) \in \mathbf{R}^p$, where $p = (d+2)n$.
4. Arrange in order the collection of functions $\{\gamma_l^{(v_k)}(g_k, \rho_k) : k = 1, \dots, n : l = 0, \dots, 5ds\}$ in the vector $\gamma = (\gamma_1, \dots, \gamma_q) \in \mathbf{R}^q$, where $q = n(5ds + 1)$.

5. Arrange in order the collection of polynomial

$$\{\pi_l^{(v_k)}(a_k, \rho_k, 1/\rho_k; P_{ij}) : (i,j) \in I_s,\ (k,l) \in K_s\}$$

as the functions $\{\pi_{\alpha\beta}(\sigma) : \alpha = 1, \ldots, m_i, \beta = 1, \ldots, q\}$ of the variable σ.

Then vector (42) may be written as

$$b(G_{g,a}) = \left(\sum_{\beta=1}^{q} \gamma_\beta \pi_{\alpha\beta}(\sigma)\right)_{\alpha=1}^{m}.$$

Let the numbers p and q be such that $p + q \leqslant m/2$. By Lemma 9.1 for fixed $v_1, \ldots, v_n$ we have the estimate

$$\begin{aligned} &card\{sgn(b(G_{g,a})) : a_k \in D^{v_k},\ g_k \in C(\mathbf{R}),\ k = 1, \ldots, n\} \\ &\leqslant card\left\{sgn\left(\sum_{\beta=1}^{q} \gamma_\beta \pi_{\alpha\beta}(\sigma)\right)_{\alpha=1}^{m} : \gamma \in \mathbf{R}^q,\ \sigma \in \mathbf{R}^p\right\} \\ &\leqslant (4s)^p (p + q + 1)^{p+2} \left(\frac{2em}{p+q}\right)^{p+q}. \end{aligned}$$

Substitute this estimate in (43) and obtain

$$card\{sgn(\operatorname{Re} b(\mathscr{R}_n))\} \leqslant 4^n (4s)^p (p + q + 1)^{p+2} \left(\frac{2em}{p+q}\right)^{p+q}. \tag{44}$$

Set $\tau = 200d^5$. We choose the numbers n, s, p, q such that

$$\tau n \leqslant s^{d-1} \leqslant 2\tau n, \quad p = nd^2, \quad q = n(2s + d + 1), \quad c_1 s^d \leqslant m \leqslant c_2 s^d,$$

and $p + q \leqslant m/2$. A direct computation leads to

$$card\{sgn(\operatorname{Re} b(\mathscr{R}_n))\} \leqslant 2^{c_0 m/2}.$$

Lemma 8.2 is proved. □

10. Uncited references

[5,9,10,12,23,24,26,31,34,39,40,44,45,49]

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Appendix A

We discuss some well-known results pertaining to orthogonal polynomials which we use in the current work.

A.1. The Gegenbauer polynomials

The Gegenbauer polynomials are usually defined via the generating function

$$(1-2tz+z^2)^{-\lambda}=\sum_{k=0}^{\infty} C_k^{\lambda}(t)z^k,$$

where $|z|<1$, $|t|<1$, and $\lambda>0$. The coefficients $C_k^{\lambda}(t)$ are algebraic polynomials of degree k and are termed the Gegenbauer polynomials associated with λ.

The Gegenbauer polynomials possess the following properties:

The family of polynomials $\{C_k^{\lambda}\}$ is a complete orthogonal system for the weighted space $L_2(I,w)$, $I=[-1,1]$, $w(t):=w_{\lambda}(t):=(1-t^2)^{\lambda-1/2}$ and

$$\begin{aligned}&\int_I C_m^{\lambda}(t)C_n^{\lambda}(t)w(t)\,dt\\ &\quad=\begin{cases}0, & m\neq n\\ v_{n,\lambda}, & m=n\end{cases}, \textit{with } v_{n,\lambda}:=\frac{\pi^{1/2}(2\lambda)_n\Gamma(\lambda+1/2)}{(n+\lambda)n!\Gamma(\lambda)},\end{aligned}\tag{A.1}$$

where we use the usual notation $(a)_0:=0$, $(a)_n:=a(a+1)\cdots(a+N-1)$.

A.2. An orthogonal system of polynomials on the sphere

We state some facts (see [43,48]) from the theory of harmonic analysis on the sphere. Let s be any positive integer. Consider the space H_s^{hom} consisting of the homogeneous harmonic polynomials of degree s in the d variables $x_1,\dots,x_d$. Any polynomial from H_s^{hom} is decomposable as a linear combination of polynomials of the form

$$h_{sk}(x)=A_{sk}\prod_{j=0}^{d-2} r_{d-j}^{k_j-k_{j-1}+1}C_{k_j-k_{j+1}}^{\frac{d-j-2}{2}+k_{j+1}}\left(\frac{x_{d-j}}{r_{d-j}}\right)(x_2\pm ix_1)^{k_{d-2}},\tag{A.2}$$

where $r_{d-j}^2=x_1^2+\cdots+x_{d-j}^2$. The vector k with integer coordinates belongs to the set

$$\begin{aligned}K^s=\{k=(k_0,k_1,\dots,k_{d-3},\varepsilon k_{d-2}):0\leqslant k_{d-2}\\ \leqslant\cdots\leqslant k_1\leqslant k_0=s,\ \varepsilon=\pm1\},\end{aligned}$$

and A_{sk} is the normalization factor

$$A_{sk} = \frac{1}{\Gamma(d/2)} \prod_{j=0}^{d-3} \times \frac{2^{2k_{j+1}+d-j-4}(k_j - k_{j+1})(d-j+2k_j-2)\Gamma^2(\frac{d-j-2}{2}+k_{j+2})}{\sqrt{\pi}\Gamma(k_j+k_{j+1}+d-j-2)}.$$

It is known that the dimension of the space H_s^{hom} is given by

$$\dim H_s^{\text{hom}} = |K^s| = \binom{s+d-1}{s} - \binom{s+d-3}{s-2}, \tag{A.3}$$

if $s \geqslant 2$, and $\dim H_0^{\text{hom}} = 1$, $\dim H_1^{\text{hom}} = d$. It is easy to verify that the dimension of H_s^{hom} is asymptotically given by

$$\dim H_s^{\text{hom}} = \left(2 + \frac{2}{(d-2)!} + c(s,d)\right) s(s+1)\cdots(s+d-3) \asymp s^{d-2}, \tag{A.4}$$

where $0 \leqslant c(s,d) \leqslant 1$ is some function depending only on s and d.

The family of functions $\{h_{sk}\}_{k \in K^s}$ is an orthonormal system in the space $L_2(S^{d-1})$, i.e., for any multi-indices $k, k' \in K^s$, the following holds

$$(h_{sk}, h_{sk'}) = \int_{S^{d-1}} h_{sk}(\xi)\overline{h_{sk'}(\xi)}\, d\xi = \delta_{kk'}.$$

Note that the spaces H_s^{hom} and $H_{s'}^{\text{hom}}$ for $s \neq s'$ are orthogonal space on S^{d-1}. The family of functions $\bigcup_{s=0}^{\infty} \{h_{sk}\}_{k \in K^s}$ is a complete orthonormal system in the space $L_2(S^{d-1})$.

The set of polynomials on the sphere $\{p : p \in \mathscr{P}_n\}$ of degree $\leqslant n$ belongs to the space $\mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_n$, which is the direct sum of the orthogonal subspaces $H_0^{\text{hom}}, H_1^{\text{hom}}, \ldots, H_n^{\text{hom}}$. From the above it follows that for any polynomial $p \in \mathscr{P}_n$ and for any function $h \in H_{n+1}^{\text{hom}} \oplus H_{n+2}^{\text{hom}} \oplus \cdots$ the equality

$$\int_{S^{d-1}} p(\xi)\overline{h(\xi)}\, d\xi = 0. \tag{A.5}$$

holds.

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